Math 1020 Week 9

Power Series

A power series is an expression of the form

$$p(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

eg Let $p(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

Put $x = \frac{1}{2}$ and 2,

$$p\left(\tfrac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 \quad \text{(convergent)}$$

$$p(2) = \sum_{n=0}^{\infty} 2^n = \infty \quad \text{(divergent)}$$

$\therefore$ $p(x)$ is convergent at $x = \frac{1}{2}$

divergent at $x = 2$

Q For what $x \in \mathbb{R}$ does $\sum_{n=0}^{\infty} c_n x^n$ converge?

Thm Suppose $p(x) = \sum_{n=0}^{\infty} c_n x^n$. Then

① There exists R, where $R \geq 0$ or ∞

such that $p(x) \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R \end{cases}$

R is called the radius of convergence of $p(x)$.

② $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ if the limit exists or $= \infty$

Rmk
- $p(x)$ may or may not converge at $x = \pm R$
- The ratio is $\left|\frac{c_n}{c_{n+1}}\right|$ above, $\left|\frac{a_{n+1}}{a_n}\right|$ in ratio test
- ② can be proved from ratio test:

Key step: Let $a_n = c_n x^n$. Then $p(x) = \sum_{n=0}^{\infty} a_n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1} x^{n+1}}{c_n x^n}\right| = \left|\frac{c_{n+1}}{c_n}\right| |x| \quad \text{if } a_n \neq 0$$

$$\boxed{R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| \text{ if the limit exists or } = \infty}$$

eg Find radius of convergence.

① $p(x) = \sum_{n=0}^{\infty} n x^n = x + 2x^2 + 3x^3 + \cdots$

Sol $c_n =$ coefficient of $x^n = n$

$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 1$$

$\therefore$ Radius of convergence $R = 1$

Rmk $R = 1 \Rightarrow p(x) \begin{cases} \text{converges if } |x|<1, \\ \text{diverges if } |x|>1 \end{cases}$

How about $x = \pm 1$?

$p(1) = \sum_{n=0}^{\infty} n$, $p(-1) = \sum_{n=0}^{\infty} (-1)^n n$ diverges by n-th term test (by n-th term test)

$\therefore$ Interval of convergence (where $p(x)$ converges) is $(-1, 1)$

② $p(x) = \sum_{n=0}^{\infty} (5^n - 2^{3n+1}) x^n$

Sol $c_n = 5^n - 2^{3n+1}$

$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{5^n - 2^{3n+1}}{5^{n+1} - 2^{3(n+1)+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5^n - 2\cdot 8^n}{5^{n+1} - 2\cdot 8^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(\frac{5}{8}\right)^n - 2}{5\cdot\left(\frac{5}{8}\right)^n - 16}\right|$$

$$= \left|\frac{0 - 2}{5\cdot 0 - 16}\right|$$

$$= \frac{1}{8}$$

$\therefore$ Radius of convergence $R = \frac{1}{8}$

Operations on Power Series

We can perform $+, -, \times,$ composition on power series, like polynomials

eg $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

Find ① $f(x) - f(-x)$

② $(2-x) f(x+x^2)$ up to x^3 term

Sol ①

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

$$f(-x) = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\therefore f(x) - f(-x) = 2x + 2x^3 + 2x^5 + \cdots$$

test.

$$f(x+x^2)$$

$$= 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$$

$$= 1 + x + x^2 + (x^2 + 2x^3 + x^4) + (x^3 + \cdots) + \cdots$$

$$= 1 + x + 2x^2 + 3x^3 + \cdots$$

(terms of x^4 or above)

$$\therefore (2-x) f(x+x^2)$$

$$= (2-x)(1 + x + 2x^2 + 3x^3 + \cdots)$$

$$= 2 - x + 2x + 4x^2 - x^2 + 6x^3 - 2x^3 + \cdots$$

$$= 2 + x + 3x^2 + 4x^3 + \cdots$$

eg

$$g(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$h(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

Find ① Radii of convergence of $g(x)$ and $h(x)$

② $g(x)\cdot g(x)$ and $g(h(x))$ up to x^3 term.

Sol For $g(x)$, $C_n=\frac{1}{n!}$

$$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|$$

$$=\lim_{n\to\infty}(n+1)=+\infty$$

$\therefore$ $R=+\infty$ (i.e. $g(x)$ converges for any x)

For $h(x)$, $C_n=\frac{(-1)^{n-1}}{n}$ $\quad |C_n|=\frac{1}{n}$

$$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$=\lim_{n\to\infty}\frac{n+1}{n}$$

$$=\lim_{n\to\infty}1+\frac{1}{n}=1$$

$\therefore$ $R=1$ ($h(x)$ converges for $|x|<1$, diverges for $|x|>1$)

$$g(x)\cdot g(x)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$+ x + x^2 + \frac{x^3}{2} + \cdots$$

$$+ \frac{x^2}{2} + \frac{x^3}{2} + \cdots$$

$$+ \frac{x^3}{6} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$$

$$g\circ h(x) = g(h(x))$$

$$= 1 + h(x) + \frac{1}{2}h(x)^2 + \frac{1}{6}h(x)^3 + \cdots$$

$$= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \frac{1}{2}\left(x - \frac{x^2}{2} + \cdots\right)^2$$

$$+ \frac{1}{6}\left(x - \cdots\right)^3 + \cdots$$

$$= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \frac{1}{2}\left(x^2 - x^3 + \cdots\right)$$

$$+ \frac{1}{6}\left(x^3 + \cdots\right) + \cdots$$

$$= 1 + x + 0x^2 + 0x^3 + \cdots$$

Binomial Theorem

Recall: $0! = 1$

For $n \geq 1$, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$

Def (Binomial coefficients)

For integers $n \geq r \geq 0$, define

$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1}$$

Rmk

- C^n_r is an integer
- C^n_r is read as "n choose r"
- C^n_r = Number of ways of choosing r objects from n distinct objects

Other notation: $C^n_r = {}_nC_r = \binom{n}{r}$

Binomial Theorem

$$\boxed{(a+b)^n = \sum_{r=0}^{n} C^n_r a^{n-r} b^r \quad \text{for } n \geq 0}$$

Why? $(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\cdots(a+b)}_{n \text{ factors}}$

To obtain $a^{n-r}b^r$, choose b from r of the n factors
a from remaining n-r factors.

$\Rightarrow$ Coefficient of $a^{n-r}b^r$ is C^n_r

eg Expand $(3-x)^4$

Sol Note $C^4_r = 1, 4, 6, 4, 1$ for $r = 0, 1, 2, 3, 4$

$$\therefore (3-x)^4 = \sum_{r=0}^{4} C^4_r (3)^{4-r}(-x)^r$$

$$= (1)(3)^4 + (4)(3)^3(-x) + 6(3)^2(-x)^2$$

$$+ (4)(3)^1(-x)^3 + (1)(-x)^4$$

$$= 81 - 108x + 54x^2 - 12x^3 + x^4$$

Formula

$C^n_0 = C^n_n = 1 \qquad C^n_1 = C^n_{n-1} = n$

$C^n_r = C^n_{n-r} \qquad C^n_r = C^{n-1}_{r-1} + C^{n-1}_r$

Pf of $C^n_r = C^{n-1}_{r-1} + C^{n-1}_r$

$$\begin{aligned}
\text{R.H.S.} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-1-r)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\
&= \frac{(n-1)!}{(r-1)!(n-1-r)!} \cdot \frac{r+n-r}{(n-r)r} \\
&= \frac{n!}{r!(n-r)!} \\
&= C^n_r \\
&= \text{L.H.S.}
\end{aligned}$$

Pascal's Triangle

					1						Row 0
				1		1					Row 1
			1		2		1				Row 2
		1		3		3		1			
	1		4		6		4		1		
1		5		10		10		5		1	

Row n is

$\underbrace{C^n_0 \; C^n_1 \; \dots \; C^n_n}_{n+1 \text{ terms}}$

$C^n_0 = C^n_n = 1 \quad \Rightarrow \quad \underline{1}$ on two ends

$C^n_r = C^{n-1}_{r-1} + C^{n-1}_r \quad \Rightarrow \quad$ each other term

= sum of two terms above it

eg To find row 6 from row 5 :

add		1		5		10		10		5		1	Row 5
	1		6		15		20		15		6		1	Row 6

$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

eg. Find constant terms and coefficients of x^3 and x^4 in $\left(\sqrt{x}-\frac{2}{x}\right)^{12}$

Sol

$$\left(\sqrt{x}-\frac{2}{x}\right)^{12}=\sum_{r=0}^{12} C_r^{12}(\sqrt{x})^{12-r}\left(-\frac{2}{x}\right)^r$$

$$=\sum_{r=0}^{12} C_r^{12} x^{6-\frac{r}{2}}(-2)^r x^{-r}$$

$$=\sum_{r=0}^{12} C_r^{12}(-2)^r \underbrace{x^{6-\frac{3}{2}r}}$$

degree $= 6-\frac{3}{2}r$

For constant term: $6-\frac{3}{2}r=0 \Rightarrow r=4$

$\therefore$ Constant term is

$$C_4^{12}(-2)^4=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}\cdot 2^4$$

$$=12\cdot 11\cdot 10\cdot 3\cdot 2$$

$$=7920$$

For x^3 term: $6-\frac{3}{2}r=3 \Rightarrow r=2$

Coefficient of $x^3 = C_2^{12}(-2)^2=\frac{12\cdot 11}{2\cdot 1}\cdot 4=264$

For x^4 term: $6-\frac{3}{2}r=4 \Rightarrow r=\frac{4}{3}$ (No such term!)

$\because r=0,1,2,\cdots,12$

$\therefore$ Coefficient of $x^4=0$

eg Find the coefficient of ab^3c^4 in $(a+b+c)^8$

Sol $(a+b+c)=\sum_{r=0}^{8} C_r^8 a^{8-r}(b+c)^r$

$$=\sum_{r=0}^{8} C_r^8 a^{8-r}\left(\sum_{s=0}^{r} C_s^r b^{r-s}c^s\right)$$

$$=\sum_{r=0}^{8}\sum_{s=0}^{r} C_r^8 C_s^r a^{8-r}b^{r-s}c^s$$

For the term ab^3c^4 $\begin{cases} 8-r=1 \\ r-s=3 \\ s=4 \end{cases} \Rightarrow s=4,\ r=7$

Coefficient of $ab^3c^4 = C_r^8 C_s^r = C_7^8 C_4^7=\frac{8!}{1!\,3!\,4!}=280$